Balanced Network, Mean Field Arguments

Basic mean field argument: N E and I neurons. Assume random connectivity: prob. connection $= K/N$, connected neurons have strength (abs value) J^{XY}. Input to neuron is

$$I_i^X = \sum_j J_{ij}^{XE} r_E - J_{ij}^{XI} r_I + h_X^i \tag{1}$$

(h_x^i can incorporate threshold, $h - \theta$; so it is negative for subthreshold external input).

Then assuming inputs fire independently and as Poisson processes at given rates, with rates uncorrelated with weights $\langle Jr \rangle = \langle J \rangle \langle r \rangle$ (all true in large-N limit). Consider updates in time bins of size 1, input is integrated over time bin. Letting mean rates per time bin be m_E, m_I, mean external input h_X, get mean input

$$\mu_X \equiv \left\langle I^X \right\rangle = K\left(J^{XE} m_E - J^{XI} m_I\right) + h_X \tag{2}$$

Spikes of type X received in timebin is Poisson process with mean Km_x, variance Km_x, so

$$\sigma_X^2 \equiv (\delta I^x)^2 = K\left(J^{XE}\right)^2 m_E + K\left(J^{XI}\right)^2 m_I \tag{3}$$

For simplicity we've ignored (1) variance in nonzero weights (2) time variance in external inputs.

(Given equation for rates in terms of inputs, and Gaussian assumption on $I's$, get equation for m_E, m_I in terms of μ's, σ's)

Basic conundrum: we believe neural noise indicates neurons fire on fluctuations. How to get mean and variance to both be $O(1)$ given $K \gg 1$? (think: distance rest to threshold $= 1$). If J's scale as $1/K$, mean is order 1 but variance is order $1/K$; if J's scale as $1/\sqrt{K}$, variance is order 1 but mean is order $\sqrt{K}$.

Some ways out:

1. Neurons not really uncorrelated; can enhance variance for given mean.

2. K is not really big: real factor is $Km\tau$. If PSP's have time constant τ, $V(t) = Je^{-(t-t_{sp})/\tau}$, mean voltage is $\bar{V} = JKm\tau$, variance is $J^2 Km\tau/2$, standard deviation is $\frac{J}{\sqrt{2}}\sqrt{Km\tau} = \bar{V}/\sqrt{2Km\tau}$. Say $\tau = 10$ms, so denominator is $\sqrt{Km/50\text{Hz}}$. For $Km = 1000 - 2000$Hz, this is $\sqrt{20-40}$, i.e. $4.5 - 6.3$. Significant but not an order of magnitude.

   E.g. if assume $J = 0.5$mV, find what's in table below. And we've neglected additional sources of variance (temporal variance of inputs; variance of nonzero weights; neuronal correlations).

Km	mean	stdev
1000	5 mV	1.1 mV
2000	10 mV	1.6 mV
4000	20 mV	2.2 mV

Nonetheless suppose we take dilemna seriously: balanced network. Discovery (VV and Somp, 1996/1998): if $J \sim 1/\sqrt{K} \rightarrow KJ \sim \sqrt{K}$ and $h \sim \sqrt{K}$, then under simple conditions network will find "balanced" solution: mean of order 1 (and variance automatically of order 1 with this scaling).

Write J as $J/\sqrt{K}$, h as $h\sqrt{K}$ with new J's, h's of order 1. To lowest order in K: solution requires

$$\boldsymbol{\mu} = \sqrt{K}(\mathbf{Jm} + \mathbf{h}) = O(1) \tag{4}$$

To lowest order in K,

$$\mathbf{Jm} + \mathbf{h} = 0 \rightarrow \mathbf{m} = -\mathbf{J}^{-1}\mathbf{h} = \frac{1}{\det J}\begin{pmatrix} J_{II} & -J_{EI} \\ J_{IE} & -J_{EE} \end{pmatrix}\begin{pmatrix} h_E \\ h_I \end{pmatrix} \equiv \frac{1}{\det J}\begin{pmatrix} \Omega_E \\ \Omega_I \end{pmatrix} \tag{5}$$

where $\Omega_E = J_{II}h_E - J_{EE}h_I$, $\Omega_I = J_{EI}h_E - J_{EE}h_I$. Rates $\mathbf{m}$ are linear in inputs and are $O(1)$ (h/J). Then residual corrections to $\mathbf{m}$ of order $1/\sqrt{K}$ give $O(1)$ contribution to input which accounts for this $O(1)$ response.

Require positive rates, so need $\det J > 0$ and $\Omega_E, \Omega_I > 0$ or all < 0.[1]

Assume m's saturate at 0, maximal value for large negative or positive inputs. Want to avoid such unbalanced solutions. Look for solutions with $m_E = 0 \rightarrow m_I = h_I/J_{II}$ to leading order, so $\mu_E = \sqrt{K}(h_E - J_{EI}h_I/J_{II}) < 0$ or $\sqrt{K}\Omega_E/J_{II} < 0$, *i.e.* this requires $\Omega_E < 0$. (Can't have $m_I < 0$ if $m_E \neq 0$, unless $\mathbf{h}_I < 0$; but threshold is order 1, $\mathbf{h}$ of order $\sqrt{K}$, so h's are > 0.)

What about solution with $m_E = m_I = m_{\max}$? Then $\mu_X = \sqrt{K}m_{max}\left(J_{EX} - J_{IX} + h_X/m_{max}\right)$ must be > 0 and $O(\sqrt{K})$. This requires $J_{IE} < J_{EE} + h_X/m_{max}$, $J_{II} < J_{EI} + h_X/m_{max}$. Can

[1] Note meaning of Ω's: can rewrite net input as

$$\mu_E = \sqrt{K}h_E\left(\frac{J_{EE}}{h_E}m_E - \frac{J_{EI}}{h_E}m_I + 1\right) \tag{6}$$

$$\mu_I = \sqrt{K}h_I\left(\frac{J_{IE}}{h_I}m_E - \frac{J_{II}}{h_I}m_I + 1\right) \tag{7}$$

$$= \sqrt{K}h_I\left(\left(\frac{\Omega_I}{h_E h_I} + \frac{J_{EE}}{h_E}\right)m_E - \left(\frac{\Omega_E}{h_E h_I} + \frac{J_{EI}}{h_E}\right)m_I + 1\right) \tag{8}$$

(9)

So $\Omega_E > 0$ means, relative to FF input, inhibition is stronger to I than to E, while $\Omega_I > 0$ means the same for excitation.

eliminate this solution at $\mathbf{h} = 0$ if $J_{IE} > J_{EE}$. Then if $m_I = m_{\max}$, $m_E = (J_{EI} m_{\max}) / J_{EE}$, so

$$\begin{aligned}\mu_I &= \sqrt{K}\left(\frac{J_{IE}J_{EI}}{J_{EE}} m_{\max} - J_{II} m_{\max}\right) && (10)\\ &= \sqrt{K}\frac{\det \mathbf{J}}{J_{EE}} m_{\max} && (11)\end{aligned}$$

which is consistent, so doesn't seem to eliminate possibility of I saturating. Also not clear unbalanced solutions are eliminated for $\mathbf{h} > 0$. They claim $J_{IE} > J_{EE}$ eliminates all unbalanced solutions, but not clear to me.

At any rate, they claim only solution is balanced solution for $\det J > 0$, $\Omega_E > 0$, $\Omega_I > 0$, $J_{IE} > J_{EE}$.

Stability

Suppose equation of form

$$\tau_X \frac{d}{dt} m_X(t) = -m_x(t) + f(\mu_k) \tag{12}$$

Then local stability given by

$$\tau_X \frac{d}{dt} \delta m_X = -\delta m_x + \sqrt{K} f'(\mu_k) \mathbf{J} \delta m_X \tag{13}$$