Balanced Network, Mean Field Arguments

Basic mean field argument: $N$ E and I neurons. Assume random connectivity: prob. connection = $K/N$, connected neurons have strength (abs value) $J_{XY}$. Input to neuron is

$$I^X_i = \sum_j J_{ij}^{XE} r^E_j - J_{ij}^{XI} r^I_i + h^X_i$$

($h^X_i$ can incorporate threshold, $h - \theta$; so it is negative for subthreshold external input).

Then assuming inputs fire independently and as Poisson processes at given rates, with rates uncorrelated with weights $\langle J r \rangle = \langle J \rangle \langle r \rangle$ (all true in large-$N$ limit). Consider updates in time bins of size 1, input is integrated over time bin. Letting mean rates per time bin be $m_E, m_I$, mean external input $h^X$; get mean input

$$\mu^X \equiv \langle I^X \rangle = K \left( J^{XE} m^E - J^{XI} m^I \right) + h^X$$

Spikes of type X received in timebin is Poisson process with mean $K m_x$, variance $K m_x$, so

$$\sigma^2_X \equiv (\delta I^x)^2 = K \left( J^{XE} \right)^2 m^E + K \left( J^{XI} \right)^2 m^I$$

For simplicity we’ve ignored (1) variance in nonzero weights (2) time variance in external inputs.

(Given equation for rates in terms of inputs, and Gaussian assumption on $I'$s, get equation for $m^E, m^I$ in terms of $\mu$’s, $\sigma$’s)

Basic conundrum: we believe neural noise indicates neurons fire on fluctuations. How to get mean and variance to both be $O(1)$ given $K \gg 1$? (think: distance rest to threshold = 1). If $J$’s scale as $1/K$, mean is order 1 but variance is order $1/K$; if $J$’s scale as $1/\sqrt{K}$, variance is order 1 but mean is order $\sqrt{K}$.

Some ways out:

1. Neurons not really uncorrelated; can enhance variance for given mean.

2. $K$ is not really big: real factor is $K m \tau$. If PSP’s have time constant $\tau$, $V(t) = J e^{-(t-t_{sp})/\tau}$, mean voltage is $\bar{V} = J K m \tau$, variance is $J^2 K m \tau / 2$, standard deviation is $\sqrt{\frac{J^2}{2} K m \tau} = \bar{V} / \sqrt{2 K m \tau}$. Say $\tau = 10$ms, so denominator is $\sqrt{K m / 50}$Hz. For $K m = 1000 - 2000$Hz, this is $\sqrt{20} - 40$, i.e. $4.5 - 6.3$. Significant but not an order of magnitude.

E.g. if assume $J = 0.5$mV, find what’s in table below. And we’ve neglected additional sources of variance (temporal variance of inputs; variance of nonzero weights; neuronal correlations).
Km mean stdev
1000  5 mV  1.1 mV
2000  10 mV  1.6 mV
4000  20 mV  2.2 mV

Nonetheless suppose we take dilemma seriously: balanced network. Discovery (VV and Somp, 1996/1998): if $J \sim 1/\sqrt{K} \rightarrow KJ \sim \sqrt{K}$ and $h \sim \sqrt{K}$, then under simple conditions network will find “balanced” solution: mean of order 1 (and variance automatically of order 1 with this scaling).

Write $J$ as $J/\sqrt{K}$, $h$ as $h\sqrt{K}$ with new $J$’s, $h$’s of order 1. To lowest order in $K$: solution requires

$$\mu = \sqrt{K}(Jm + h) = O(1)$$

To lowest order in $K$,

$$Jm + h = 0 \rightarrow m = -J^{-1}h = \frac{1}{\det J} \begin{pmatrix} J_{II} & -J_{EI} \\ J_{IE} & -J_{EE} \end{pmatrix} \begin{pmatrix} h_E \\ h_I \end{pmatrix} \equiv \frac{1}{\det J} \begin{pmatrix} \Omega_E \\ \Omega_I \end{pmatrix}$$

where $\Omega_E = J_{II}h_E - J_{EE}h_I$, $\Omega_I = J_{EI}h_E - J_{EE}h_I$. Rates $m$ are linear in inputs and are $O(1)$ ($h/J$). Then residual corrections to $m$ of order $1/\sqrt{K}$ give $O(1)$ contribution to input which accounts for this $O(1)$ response.

Require positive rates, so need $\det J > 0$ and $\Omega_E, \Omega_I > 0$ or all $< 0$.\(^1\)

Assume $m$’s saturate at 0, maximal value for large negative or positive inputs. Want to avoid such unbalanced solutions. Look for solutions with $m_E = 0 \rightarrow m_I = h_I/J_{II}$ to leading order, so $\mu_E = \sqrt{K}(h_E - J_{EI}h_I/J_{II}) < 0$ or $\sqrt{K}\Omega_E/J_{II} < 0$, i.e. this requires $\Omega_E < 0$. (Can’t have $m_I < 0$ if $m_E \neq 0$, unless $h_I < 0$; but threshold is order 1, $h$ of order $\sqrt{K}$, so $h$’s are $> 0$.)

What about solution with $m_E = m_I = m_{max}$? Then $\mu_X = \sqrt{K}m_{max}(J_{EX} - J_{IX} + h_X/m_{max})$ must be $> 0$ and $O(\sqrt{K})$. This requires $J_{IE} < J_{EE} + h_X/m_{max}$, $J_{II} < J_{EI} + h_X/m_{max}$. Can

\(^1\)Note meaning of $\Omega$’s: can rewrite net input as

$$\mu_E = \sqrt{K}h_E \left( \frac{J_{EE}}{h_E} m_E - \frac{J_{EI}}{h_E} m_I + 1 \right)$$

$$\mu_I = \sqrt{K}h_I \left( \frac{J_{IE}}{h_I} m_E - \frac{J_{II}}{h_I} m_I + 1 \right)$$

$$= \sqrt{K}h_I \left( \left( \frac{\Omega_I}{h_E h_I} + \frac{J_{EE}}{h_E} \right) m_E - \left( \frac{\Omega_E}{h_E h_I} + \frac{J_{EI}}{h_E} \right) m_I + 1 \right)$$

So $\Omega_E > 0$ means, relative to FF input, inhibition is stronger to I than to E, while $\Omega_I > 0$ means the same for excitation.
eliminate this solution at $h = 0$ if $J_{IE} > J_{EE}$. Then if $m_I = m_{\text{max}}$, $m_E = (J_{EI} m_{\text{max}}) / J_{EE}$, so

$$\mu_I = \sqrt{K} \left( \frac{J_{IE} J_{EI}}{J_{EE}} m_{\text{max}} - J_{II} m_{\text{max}} \right)$$  \hspace{1cm} (10)$$

$$= \sqrt{K} \frac{\det J}{J_{EE}} m_{\text{max}}$$  \hspace{1cm} (11)$$

which is consistent, so doesn’t seem to eliminate possibility of I saturating. Also not clear unbalanced solutions are eliminated for $h > 0$. They claim $J_{IE} > J_{EE}$ eliminates all unbalanced solutions, but not clear to me.

At any rate, they claim only solution is balanced solution for $\det J > 0$, $\Omega_E > 0$, $\Omega_I > 0$, $J_{IE} > J_{EE}$.

**Stability**

Suppose equation of form

$$\tau_X \frac{d}{dt} m_X(t) = -m_x(t) + f(\mu_k)$$  \hspace{1cm} (12)$$

Then local stability given by

$$\tau_X \frac{d}{dt} \delta m_X = -\delta m_x + \sqrt{K} f'(\mu_k) J \delta m_X$$  \hspace{1cm} (13)$$