Non-normal dynamics

- Linear system, all eigenvalues negative real parts: what happens to a small perturbation from the fixed point?

  Non-orthogonal eigs, big things cancelling to make something small, now imagine in 1000 dimensions, etc.

- Non-normal: $\mathbf{W}\mathbf{W}^\dagger \neq \mathbf{W}^\dagger\mathbf{W}$

  (commute iff have common basis of eigs;

  right eigs of $\mathbf{W}$ are left eigs of $\mathbf{W}^\dagger$ ($\mathbf{W}\mathbf{e} = \lambda\mathbf{e} \rightarrow \mathbf{e}^\dagger\mathbf{W}^\dagger = \lambda\mathbf{e}^\dagger$);

  so non-normal $\leftrightarrow$ right eigs $\neq$ left eigs $\leftrightarrow$ right eigs orthonormal ($\mathbf{W} = \mathbf{E}\Lambda\mathbf{E}^{-1}$, right eigs columns of $\mathbf{E}$, left eigs rows of $\mathbf{E}^{-1}$; right eigs = left eigs $\leftrightarrow$ $\mathbf{E}^\dagger = \mathbf{E}^{-1}$).

- Neurobiological connection matrices are of the form $\mathbf{W} = \begin{pmatrix} \mathbf{W}_{EE} & -\mathbf{W}_{EI} \\ \mathbf{W}_{IE} & -\mathbf{W}_{II} \end{pmatrix}$, with all entries of the $\mathbf{W}_{XY}$ being non-negative. The simplest way to see that these are non-normal is just to consider the arrangement of the signs of the nonzero entries: $\begin{pmatrix} + & - \\ + & - \end{pmatrix}$. For such a matrix, $\mathbf{W}\mathbf{W}^T$ has signs $\begin{pmatrix} + & + \\ + & + \end{pmatrix}$, while $\mathbf{W}^T\mathbf{W}$ has signs $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$. So, assuming the off-diagonal blocks are not all zero, $\mathbf{W}$ is non-normal.

A simple example

$\mathbf{W} = \begin{pmatrix} w & -kw \\ w & -kw \end{pmatrix}$, $k > 1$.

Eigs: $\mathbf{e}_1 = \frac{1}{\sqrt{1+k^2}}\begin{pmatrix} k \\ 1 \end{pmatrix}$, $\lambda_1 = 0$; $\mathbf{e}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_+ = (1-k)w$; Note $\lambda_+ < 0$.

(not orthogonal)

$\frac{d\mathbf{r}}{dt} = -\mathbf{r} + \mathbf{W}\mathbf{r} + \mathbf{h}$ ($\mathbf{r} = \begin{pmatrix} r_E \\ r_I \end{pmatrix}$)

$$\frac{dr_E}{dt} = -r_E + wr_E - kwr_I + \mathbf{h}_E \quad (1)$$

$$\frac{dr_I}{dt} = -r_I + wr_E - kwr_I + \mathbf{h}_I \quad (2)$$

Form sum and difference coordinates: $r_+ = \frac{r_E+r_I}{\sqrt{2}}$, $r_- = \frac{r_E-r_I}{\sqrt{2}}$. Note $\frac{2}{\sqrt{2}}\frac{wr_E-kwr_I}{\sqrt{2}} = w(1-$

$k)\mathbf{r}_+ + w(1+k)r_- \equiv \lambda_+ r_+ + w_{FF} r_-$.

$$\frac{dr_+}{dt} = -r_+ + \lambda_+ r + w_{FF}\mathbf{r}_- + \mathbf{h}_+ \quad (3)$$

$$\frac{dr_-}{dt} = -r_- + \mathbf{h}_- \quad (4)$$

where $h_+ = \frac{\mathbf{h}_E + \mathbf{h}_I}{\sqrt{2}}$, $h_- = \frac{\mathbf{h}_E - \mathbf{h}_I}{\sqrt{2}}$.

In orthogonal basis: $\mathbf{r}_+ = \mathbf{e}_+$, $\mathbf{r}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $r_+ = \mathbf{r}\cdot\mathbf{r}_+$, $\mathbf{r}_- = \mathbf{r}\cdot\mathbf{r}_-$. So this is equation for components in this basis.

DRAW PICTURE: feedforward connectivity (w/ one nonzero recurrent connection); AMPLIFY W/O SLOWING

Behavior: consider decay from perturbation of r_E: $\mathbf{h} = 0$, $r_+(0) = r_-(0)$ ($r_E(0) > 0$, $r_I(0) = 0$).

$$\mathbf{r}_-(t) = r_-(0)e^{-t} \quad (5)$$

$$\mathbf{r}_+(t) = r_+(0)e^{-(1-\lambda_+)t} + w_{FF}\int_0^t e^{-(1-\lambda_+)(t-t')} r_-(t') \quad (6)$$

$$\int_0^t dt\, e^{-(1-\lambda_+)(t-t')} r_-(t) = r_-(0)\int_0^t dt\, e^{-(1-\lambda_+)(t-t')} e^{-t'} \quad (7)$$

$$= r_-(0)e^{-(1-\lambda_+)t}\frac{e^{-\lambda_+ t} - 1}{-\lambda_+} \quad (8)$$

So

$$\mathbf{r}_+(t) = r_+(0)e^{-(1-\lambda_+)t} + \frac{w_{FF} r_-(0)}{-\lambda_+}\left(e^{-t} - e^{-(1-\lambda_+)t}\right) \quad (9)$$

INTERPRETATION: SMALL DIFFS $\rightarrow$ LARGE SUM RESPONSES

Consider initial condition = 0, turn on $h_+ = h_-$ at $t = 0$:

$$\mathbf{r}_-(t) = h_-\left(1 - e^{-t}\right) \quad (10)$$

$$\mathbf{r}_+(t) = \frac{h_+}{1-\lambda_+}\left(1 - e^{-(1-\lambda_+)t}\right) + w_{FF}\int_0^t e^{-(1-\lambda_+)(t-t')} r_-(t') \quad (11)$$

$$= \frac{h_+}{1-\lambda_+}\left(1 - e^{-(1-\lambda_+)t}\right) + w_{FF}h_- e^{-(1-\lambda_+)t}\left(\frac{e^{(1-\lambda_+)t} - 1}{1-\lambda_+} - \frac{e^{-\lambda_+ t} - 1}{-\lambda_+}\right) \quad (12)$$

$$= \frac{h_+}{1-\lambda_+}\left(1 - e^{-(1-\lambda_+)t}\right) + w_{FF}h_- \frac{-\lambda_+ + e^{-(1-\lambda_+)t} - (1-\lambda_+)e^{-t}}{(-\lambda_+)(1-\lambda_+)} \quad (13)$$

$$= (w_{FF}h_- + h_+)\frac{1 - e^{-(1-\lambda_+)t}}{1-\lambda_+} - w_{FF}h_- \frac{e^{-t} - e^{-(1-\lambda_+)t}}{-\lambda_+} \quad (14)$$

SHOW SLIDES

Schur Transformation, General Case

Order eigenvalues (*e.g.*, from max to min real part)

Apply Gram-Schmidt orthonormalization to obtain $\mathbf{s}_1 = \mathbf{e}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N$ – orthonormal Schur basis vectors.

Then eigenvectors can be written

$$\mathbf{e}_1 = (\mathbf{s}_1 \cdot \mathbf{e}_1)\mathbf{s}_1 \tag{15}$$

$$\mathbf{e}_2 = (\mathbf{s}_1 \cdot \mathbf{e}_2)\mathbf{s}_1 + (\mathbf{s}_2 \cdot \mathbf{e}_2)\mathbf{s}_2 \tag{16}$$

$$\ldots \tag{17}$$

or

$$(\mathbf{e}_1 \ \mathbf{e}_2 \ \ldots \ \mathbf{e}_n) = (\mathbf{s}_1 \ \mathbf{s}_2 \ \ldots \ \mathbf{s}_n)\,(\text{upper triangular}) \tag{18}$$

i.e. $\mathbf{E} = \mathbf{QR}$ with $\mathbf{Q}$ unitary and $\mathbf{R}$ upper triangular. Then $\mathbf{W} = \mathbf{E}\Lambda\mathbf{E}^{-1} = \mathbf{QR}\Lambda\mathbf{R}^{-1}\mathbf{Q}^{\dagger} = \mathbf{QSQ}^{\dagger}$. $\mathbf{S} = \mathbf{R}\Lambda\mathbf{R}^{-1}$ is the (upper triangular) matrix in the Schur representation, with the eigenvalues on the diagonals.

GENERAL PICTURE: eigenvector representation, Schur representation, Schur rep with balance $\rightarrow$ small eigenvalues = feedforward network.

Schur basis is NOT UNIQUE. But, sum of squares of FF weights = $\mathrm{Tr}\, WW^{\dagger} - \sum_i |\lambda_i|^2$ is a unitary invariant.

DON'T CONFUSE WITH JORDAN FORM. Jordan form is *similarity* transform to almost-diagonal given (non-generic) missing eigenvector.

Schur basis is *unitary* transformation for any matrix (generically diagonizable with similarity transformation).

Example: submatrices simultaneously diagonizable by unitary transform

e.g. if translation invariant: $\rightarrow 2 \times 2$ submatrices for each freq.

Example: Rajan and Abbott

Example: $\begin{pmatrix} \mathbf{W}_E & -\mathbf{W}_I \\ \mathbf{W}_E & -\mathbf{W}_I \end{pmatrix}$ go to sum and diff basis of those eigs; sum modes have eigs $\lambda_i^E - \lambda_i^I$; also N zero eigs; diff modes give $\lambda_i^E + \lambda_i^I$ times sum modes.

SLIDE

Schur Basis, 2×2 case

Eigenvalues:

$$\lambda_{\pm} = \frac{w_{EE} - w_{II}}{2} \pm \sqrt{\left(\frac{w_{EE} + w_{II}}{2}\right)^2 - w_{EI} w_{IE}} \tag{19}$$

$$\equiv x_- \pm \sqrt{x_+^2 - w_{EI} w_{IE}} \tag{20}$$

Eigenvectors:

$$\mathbf{e}_{\pm} \propto \begin{pmatrix} 1 \\ (\lambda_{\mp} + w_{II})/w_{EI} \end{pmatrix} = \begin{pmatrix} 1 \\ \left(x_+ \mp \sqrt{x_+^2 - w_{EI} w_{IE}}\right)/w_{EI} \end{pmatrix} \tag{21}$$

Take $|\mathbf{e}_{\pm}| = 1$. Note: both elements have real parts > 0; sum mode. Assume eigenvectors are distinct.

Let Schur basis be $\{\mathbf{e}_-, \mathbf{q}\}$. $\mathbf{q} \propto \begin{pmatrix} (\lambda_+^* + w_{II})/w_{EI} \\ -1 \end{pmatrix}$: difference mode.

Gram-Schmidt:

$$\mathbf{q} = \frac{\mathbf{e}_+ - (\mathbf{e}_- \cdot \mathbf{e}_+)\mathbf{e}_-}{\sqrt{1 - |\mathbf{e}_- \cdot \mathbf{e}_+|^2}} \tag{22}$$

Then compute

$$\mathbf{W}\mathbf{q} = \frac{\lambda_+ \mathbf{e}_+ - \lambda_-(\mathbf{e}_- \cdot \mathbf{e}_+)\mathbf{e}_-}{\sqrt{1 - |\mathbf{e}_- \cdot \mathbf{e}_+|^2}} = \lambda_+ \mathbf{q} + \frac{(\lambda_+ - \lambda_-)(\mathbf{e}_- \cdot \mathbf{e}_+)}{\sqrt{1 - |\mathbf{e}_- \cdot \mathbf{e}_+|^2}} \mathbf{e}_- \tag{23}$$

Thus, in the Schur basis $(\mathbf{e}_-, \mathbf{q})$, $\mathbf{W}$ takes the upper triangular form

$$\mathbf{W} = \begin{pmatrix} \lambda_- & \beta \\ 0 & \lambda^+ \end{pmatrix} \text{ with}$$

$$\beta = \frac{(\lambda_+ - \lambda_-)(\mathbf{e}_- \cdot \mathbf{e}_+)}{\sqrt{1 - |\mathbf{e}_- \cdot \mathbf{e}_+|^2}} \tag{24}$$

β is the effective feedforward weight from the difference mode $\mathbf{q}$ to the sum mode $\mathbf{e}_+$.

General matrix, distinct eigenvectors $\mathbf{e}_1, \mathbf{e}_2$, eigenvalues λ_1, λ_2: β is

- Large when $|\mathbf{e}_1 \cdot \mathbf{e}_2|$ is close to one, i.e. when the angle between the eigenvectors is small;
- Zero when the matrix is normal, so that $|\mathbf{e}_1 \cdot \mathbf{e}_2| = 0$.

(It also becomes zero if $\lambda_1 = \lambda_2$, but this means that the matrix is normal, assuming there are two distinct eigenvectors.)

Compute β:

- Eigs real: $x_+^2 - w_{EI}w_{IE} > 0$

$$\begin{aligned} |\beta|^2 &= \mathrm{Tr}\,\mathbf{W}\mathbf{W}^\dagger - |\lambda_+|^2 - |\lambda_-|^2 & (25)\\ &= w_{EE}^2 + w_{EI}^2 + w_{IE}^2 + w_{II}^2 - 2(x_-^2 + x_+^2 - w_{EI}w_{IE}) & (26)\\ &= (w_{EI} + w_{IE})^2 & (27) \end{aligned}$$

Note $\beta > 0$, therefore $\beta = w_{EI} + w_{IE}$. With real eigs, β is large when $\mathbf{W}$ deviates strongly from Hermitian.

- Eigs complex: $w_{EI}w_{IE} - x_+^2 > 0$

$$\begin{aligned} |\beta|^2 &= \mathrm{Tr}\,\mathbf{W}\mathbf{W}^\dagger - |\lambda_+|^2 - |\lambda_-|^2 & (28)\\ &= w_{EE}^2 + w_{EI}^2 + w_{IE}^2 + w_{II}^2 - 2(x_-^2 + w_{EI}w_{IE} - x_+^2) & (29)\\ &= (w_{EI} - w_{IE})^2 + (w_{EE} + w_{II})^2 & (30) \end{aligned}$$

With complex eigs, β is large when $\mathbf{W}$ deviates strongly from anti-Hermitian.

General Solution, 2×2 case

$$\tau \frac{d\mathbf{r}}{dt} = -\mathbf{r} + \mathbf{W}\mathbf{r} + \mathbf{h} \tag{31}$$

Solution:

$$\mathbf{r}(t) = e^{-(\mathbf{1}-\mathbf{W})t/\tau}\mathbf{r}(0) + \frac{1}{\tau}\int_0^t dt' e^{-(\mathbf{1}-\mathbf{W})(t-t')/\tau}\mathbf{h}(t') \tag{32}$$

$$e^{-(\mathbf{1}-\mathbf{W})t/\tau} = e^{-t/\tau}\begin{pmatrix} e^{\lambda_+ t/\tau} & \beta\frac{e^{\lambda_- t/\tau} - e^{\lambda_+ t/\tau}}{\lambda_- - \lambda_+} \\ 0 & e^{\lambda_- t/\tau} \end{pmatrix} \tag{33}$$

$$\frac{1}{\tau}\frac{e^{(\lambda_- - 1)t/\tau} - e^{(\lambda_+ - 1)t/\tau}}{\lambda_- - \lambda_+} = \frac{1}{\tau}e^{(\lambda_- - 1)t/\tau} \star \frac{1}{\tau}e^{(\lambda_+ - 1)t/\tau} \tag{34}$$

$$\text{where} \qquad f(t) \star g(t) \equiv \int_0^t f(t-t')g(t') \tag{35}$$

Generalization: the i^{th} pattern simply filters (convolves) its input with its filter, $f_i(t) = \frac{1}{\tau}e^{-(1-\lambda_i)t/\tau}$; any initial condition $r_i(0)$ is multiplied by $\tau f_i(t)$. The total input to pattern i at time t is $I_i^{total}(t) = I_i(t) + \sum_{j>i} w_{ij} r_j(t)$ where $r_j(t)$ is the activity of node j. Then node i's activity $r_i(t) = f_i \star I_i^{total}(t) + r_i(0)\tau f_i(t)$ where $\star$ indicates convolution. Start at nodes receiving no feedforward input from other patterns, compute their rates, proceed to the next higher nodes, and iterate.

Alternatively, from this, one can compute the matrix $\frac{1}{\tau}e^{-(\mathbf{1}-\mathbf{W})t/\tau}$, which gives the solution Eq. 32. By collecting all contributions of input at j to response at i from the solution just described, we can compute that the element $\frac{1}{\tau}\left(e^{-(\mathbf{1}-\mathbf{W})t/\tau}\right)_{ij}$ is the sum, over all feedforward paths from j to i, of the following for each path: the concatenation (convolutions) of the filters for each pattern in the path (including j and i), multiplied by the product of all the feedforward weights along the path. If $i = j$ (diagonal elements), this is just the filter for the node j; if there are no feedforward paths, the element is 0.

Pseudospectra

Spectra: $\exists \mathbf{v}$ s.t. $\mathbf{W}\mathbf{v} = z\mathbf{v}$ for scalar z. In terms of resolvent of $\mathbf{W}$, $R_{\mathbf{W}}(z) = (W - z\mathbf{1})^{-1}$, spectrum is $\sigma(\mathbf{W}) = \{z : ||R_{\mathbf{W}}(z)|| = \infty\}$

ϵ-Pseudospectra: $\sigma_\epsilon(\mathbf{W}) = \{z : ||R_{\mathbf{W}}(z)|| > \frac{1}{\epsilon}\}$, *i.e.* for vector-induced norm, $\exists \mathbf{v}$ s.t. $|\mathbf{W}\mathbf{v} - z\mathbf{v}| < \epsilon$.

For vector-induced norm: equivalent definition is $\sigma_\epsilon(\mathbf{W}) = \{z : \exists \mathbf{d}W, ||\mathbf{d}W|| < \epsilon \text{ s.t. } z \in \sigma(\mathbf{W} + \mathbf{d}W)\}$. For normal matrix, pseudospectrum is just the set of ϵ-balls around the spectrum. For non-normal, it always includes that but is generally larger, can be much larger – spectrum very non-robust to perturbation.

Example: $\mathbf{M} = \begin{pmatrix} \lambda_1 & \beta \\ 0 & \lambda_2 \end{pmatrix}$. Perturb with $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$, size ϵ. Spectrum is perturbed from $\{\lambda_1, \lambda_2\}$ to

$$
\begin{aligned}
& \frac{\lambda_1+\lambda_2}{2} \pm \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2 + \beta\epsilon} && (36)\\
= \quad & \frac{\lambda_1+\lambda_2}{2} \pm \frac{\lambda_1-\lambda_2}{2}\sqrt{1+\frac{4\beta\epsilon}{(\lambda_1-\lambda_2)^2}} && (37)\\
= \quad & \left\{\lambda_1 + \frac{\lambda_1-\lambda_2}{2}\left(\sqrt{1+\frac{4\beta\epsilon}{(\lambda_1-\lambda_2)^2}}-1\right),\right. && (38)\\
& \left.\lambda_2 - \frac{\lambda_1-\lambda_2}{2}\left(\sqrt{1+\frac{4\beta\epsilon}{(\lambda_1-\lambda_2)^2}}-1\right)\right\} && (39)\\
\to_{\beta\to\infty} \quad & \frac{\lambda_1+\lambda_2}{2} \pm \sqrt{\frac{\beta}{\epsilon}}\epsilon && (40)
\end{aligned}
$$

SLIDE

Kreiss Bound

Pseudospectra give a lower bound on the size of transients of a dynamics $\frac{d\mathbf{x}}{dt} = \mathbf{W}\mathbf{x}$, where all the eigenvalues of $\mathbf{W}$ have negative real part:

$$\sup_{t\geq 0} ||e^{t\mathbf{W}}|| \geq \alpha_\epsilon(\mathbf{W})/\epsilon \quad \text{for all } \epsilon > 0 \tag{41}$$

where $\alpha_\epsilon(\mathbf{W})$ is the largest real part of an element of the ϵ-pseudospectrum of $\mathbf{W}$. This leads directly to the Kreiss bound: defining

$$\mathcal{K}(\mathbf{W}) = \sup_{\epsilon>0} \alpha_\epsilon(\mathbf{W})/\epsilon = \sup_{\Re z>0} (\Re(z))||(z-\mathbf{W})^{-1}|| \tag{42}$$

then

$$\sup_{t\geq 0} ||e^{t\mathbf{W}}|| \geq \mathcal{K}(\mathbf{W}) \tag{43}$$

The bound in Eq. 41 is established as follows. There is always some real ω and $M \geq 1$ such that $||e^{t\mathbf{W}}|| \leq Me^{\omega t}$ for all $t \geq 0$. ω can be any number larger than the largest real part of an eigenvalue of $\mathbf{W}$ (you may have to choose M large to compensate for transients at finite times), so when all eigenvalues have negative real part we can always choose $\omega < 0$.

Then, for any z with $\Re z > w$, and in particular for any z with $\Re z > 0$,

$$(z-\mathbf{W})^{-1} = \int_0^\infty dt\, e^{-t(z-\mathbf{W})} = \int_0^\infty dt\, e^{-tz} e^{-t\mathbf{W}} \tag{44}$$

This implies

$$||(z-\mathbf{W})^{-1}|| \leq \int_0^\infty dt\, |e^{-tz}|\, ||e^{-t\mathbf{W}}|| \tag{45}$$

Let $S = \sup_{t\geq 0} ||e^{t\mathbf{W}}||$. Then

$$||(z-\mathbf{W})^{-1}|| \leq S \int_0^\infty dt\, |e^{-tz}| = S/(\Re(z)) \tag{46}$$

or

$$\sup_{t\geq 0} ||e^{t\mathbf{W}}|| \geq (\Re(z))||(z-\mathbf{W})^{-1}|| \tag{47}$$

How long do pseudospectra act like spectra?

Pseudospectrum can "act like eigenvalues" during transient period, if move to positive real part by more than ϵ. If z is in $\sigma_\epsilon(\mathbf{W})$, $\Re(z) > 0$, $\Re(z)/\epsilon > 1$, then for any $\tau > 0$,

$$\sup_{0<t\leq\tau} ||e^{tW}|| \geq e^{\Re(z)\tau} \Big/ \left(1 + \epsilon \frac{e^{\Re(z)\tau} - 1}{\Re(z)}\right) \tag{48}$$

Expression in paren is close to 1 for $e^{\Re(z)\tau} \ll 1 + \Re(z)/\epsilon$, i.e. $\tau < \frac{\log(1+\Re(z)/\epsilon)}{\Re(z)}$. On those time scales, z acts like an eigenvalue.